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ENUMERATION OF Q-ACYCLIC SIMPLICIAL COMPLEXES

BY
GIL KALAI

ABSTRACT

Let $\mathscr{C} = \mathscr{C}(n, k)$ be the class of all simplicial complexes C over a fixed set of n vertices ($2 \leqq k \leqq n$) such that: (1) C has a complete $(k-1)$-skeleton, (2) C has precisely $\binom{n-1}{k}$ k-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathscr{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum_{C \in \mathscr{C}} |H_{k-1}(C)|^2 = n^{\binom{n-2}{k}}.$$

This formula extends to high dimensions Cayley's formula for the number of trees on n labelled vertices. Its proof is based on a generalization of the matrix tree theorem.

1. Introduction

The purpose of this paper is to generalize Cayley's formula for the number of labelled trees, to k-dimensional simplicial complexes with a complete $(k-1)$-skeleton. The technique of proof is a generalization of the matrix tree theorem, see [2, 7]. The main result of this paper is the following:

THEOREM 1. *Let $\mathscr{C} = \mathscr{C}(n, k)$ be the class of all k-dimensional simplicial complexes C over a fixed set V of n vertices $(2 \leqq k \leqq n)$, such that*:

(1) *C has a complete $(k-1)$-skeleton*;

(2) *C has precisely $\binom{n-1}{k}$ k-faces*;

(3) $H_k(C) = 0$.

Then

$$\sum_{C \in \mathscr{C}} |H_{k-1}(C)|^2 = n^{\binom{n-2}{k}}.$$

All homology groups with unspecified coefficient ring in this paper are with integer coefficients.

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REMARK. If we set $k = 1$ in Theorem 1, then $\mathscr{C}$ becomes the set $\mathbf{T}_n$ of all trees on a fixed set of n vertices. The assertion of Theorem 1 then reduces to Cayley's formula $|\mathbf{T}_n| = n^{n-2}$, provided we use the reduced homology $\hat{H}_0(C)$ [8, p. 168], which, for a connected graph, is 0 rather than $\mathbf{Z}$. In fact, our proof of Theorem 1, applied to the case $k = 1$ with reduced homology, reduces to the proof of Cayley's formula via the matrix tree theorem.

Theorem 1 is proved in Section 3.

Further results concerning tree enumeration can be generalized in the same spirit. Some examples will be considered in Section 4, e.g.:

For a k-dimensional simplicial complex C and a face $u \in C$, define the degree $\deg_C u$ of u to be the number of k-faces of C which include u. For $V = \{x_1, \cdots, x_n\}$, define:

$$\mathscr{C}(n, k\,; d_1, \cdots, d_n) = \{C \in \mathscr{C}(n, k) : \deg_C x_i = d_i \text{ for } 1 \leqq i \leqq n\}.$$

THEOREM 3. *Put*

$$m_1 = \binom{n-2}{k-1}, \qquad m_2 = \binom{n-2}{k}.$$

Then

$$\sum \{|H_{k-1}(C)|^2 : C \in \mathscr{C}(n, k\,; d_1, \cdots, d_n)\} = \binom{m_2}{d_1 - m_1, d_2 - m_1, \cdots, d_n - m_1}.$$

In particular, $\mathscr{C}(n, k\,; d_1, \cdots, d_n) \neq \emptyset$ *iff* $d_i \geqq m_1$ *for all* i, *and* $\sum_{i=1}^{n} (d_i - m_1) = m_2$.

It would be interesting to find similar formulas when the degrees of the p-faces are specified for some fixed $p > 0$.

In Section 5 we apply Theorem 1 to show that for fixed $k > 1$, and for large n, the average of the order of $H_{k-1}(C)$ over all $C \in \mathscr{C}(n, k)$ is very large. We also discuss some open problems concerning the estimation of the number of complexes in $\mathscr{C}(n, k)$ and in some related families of complexes. In Section 6 we discuss the duality map between $\mathscr{C}(n+2, k)$ and $\mathscr{C}(n+2, n-k)$. In the final section we describe the situation for $k = 2$ and $n \leqq 6$. It turns out that for $n \leqq 6$ all members of $\mathscr{C}(n, 2)$ are collapsible, and therefore have trivial homology groups, except for the triangulations of the projective plane P^2 with six vertices that are obtained by identifying opposite faces of a regular icosahedron. Since $H_1(P^2) = \mathbf{Z}_2$, each of these complexes is counted in Theorem 1 four times.

For a simplicial complex C, $f_p(C)$ will denote the number of p-faces of C. For a face $s \in C$, the *link* of s in C, link (s, C) is defined by

$$\text{link}(s, C) = \{t \setminus s : t \in C, t \supset s\}.$$

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2. Algebraic topology preliminaries

For basic definitions and results concerning simplicial complexes and homology theory the reader should consult [6] or [8], or any other textbook on algebraic topology.

Let C be a simplicial complex and let $V = \{x_1, \cdots, x_n\}$ be the vertex set of C. Let s be a $(p-1)$-face and let t be a p-face of C. The *incidence number* $I(s, t)$ is defined as follows: If $s \not\subset t$, then $I(s, t) = 0$. If $s \subset t$, suppose $t = \{x_{i_0}, \cdots, x_{i_p}\}$ where $i_0 < i_1 < \cdots < i_p$. If $s = \{x_{i_0}, \cdots, \hat{x}_{i_j}, \cdots, x_{i_p}\}$ ($\hat{}$ means deletion), then $I(s, t) = (-1)^j$.

We regard the array $I(s, t)$ as a rectangular matrix of order $f_{p-1}(C) \times f_p(C)$, with rows corresponding to $(p-1)$-faces, and columns corresponding to p-faces. This matrix will be called the p-th incidence matrix of C, and denoted by $I^p(C)$.

REMARK. Usually it is assumed that the faces of C are oriented, and for $s \subset t$ the incidence number $I(s, t)$ is 1 if s is oriented coherently with t and -1 otherwise. Our definition coincides with this definition when the faces of C are oriented according to the order of their vertices. In fact, the results and proofs of the next sections will not depend on our concrete definition of $I(s, t)$.

Recall that the p-th chain group $C_p(C)$ $(p \geqq 0)$ is the free abelian group generated by the p-faces of C. The incidence matrix $I^p(C)$ $(p \geqq 1)$ represents the chain group homomorphism $\partial_p : C_p(C) \to C_{p-1}(C)$, i.e., the boundary operation, with respect to the standard bases of $C_p(C)$ and $C_{p-1}(C)$.

Recall that $Z_p(C)$ and $B_p(C)$, the groups of p-cycles and p-boundaries of C, are defined by $Z_p(C) = \ker \partial_p$ and $B_p(C) = \text{im}\, \partial_{p+1}$. Since $\partial_p \partial_{p+1} = 0$ (equivalently: $I_p(C) I_{p+1}(C) = 0$), $B_p(C) \subset Z_p(C)$, and the p-th homology group of C is defined by $H_p(C) = Z_p(C) / B_p(C)$. The abelian group $H_p(C)$ can be written as a sum $H_p(C) = F_p(C) \oplus T_p(C)$, where $F_p(C)$ is a free abelian group and $T_p(C)$ is a finite abelian group. $\beta_p(C) = \text{rank}\, F_p(C)$ is called the p-th Betti number of C.

C is *acyclic* if $H_0(C) = \mathbf{Z}$ (i.e., C is connected) and $H_p(C) = 0$ for all $p > 0$. C is **Q**-*acyclic* if $H_0(C, \mathbf{Q}) = \mathbf{Q}$ and $H_p(C, \mathbf{Q}) = 0$ for all $p > 0$. Equivalently, C is

Q-acyclic if $\beta_0(C)=1$ and $\beta_p(C)=0$ for all $p>0$. (Thus $H_p(C)$ is a finite group for all $p>0$.)

We shall use the following elementary result:

PROPOSITION 1. *Let K be a complete k-dimensional complex. Then $Z_k(C)$ is freely generated by the boundaries of all $(k+1)$-faces, on vertices of K, that contain a fixed vertex.*

Occasionally we shall need the concept of collapsibility of a simplicial complex (see also [4]). A face s of a simplicial complex C is *free* if s is included in a unique maximal face of C. Let C be a simplicial complex, t a maximal face of C and s a free subface of t of dimension $\dim t-1$. The operation of deleting t and s from C is called an *elementary collapse*. C is *collapsible* if it can be reduced to the void complex by a sequence of elementary collapses. If C collapses to C', then $H_k(C)=H_k(C')$ for every k. (In fact, C is homotopy equivalent to C'.) Thus, if C is collapsible it is acyclic.

3. Proof of the main theorem

For fixed $k>1$ and $n>k$, let K be the complete k-dimensional complex on the vertex set $V=\{x_1,\cdots,x_n\}$. Consider the class $\mathcal{A}=\mathcal{A}(n,k)$ of all subcomplexes of K with a complete $(k-1)$-skeleton.

For $C\in\mathcal{A}(n,k)$ define the *reduced incidence matrix* $I_r^k(C)$ to be the $\binom{n-1}{k}\times f_k(C)$ matrix obtained from $I^k(C)$ by deleting all the rows that correspond to $(k-1)$-faces which contain the vertex x_1.

Let $\mathcal{C}=\mathcal{C}(n,k)$ be the class of all members C of $\mathcal{A}(n,k)$ that satisfy any two of the following three additional conditions:

(1) $f_k(C)=\binom{n-1}{k}$ (i.e., C has $\binom{n-1}{k}$ k-faces).

(2) $H_k(C)=0$.

(3) $H_{k-1}(C)$ is a finite group.

PROPOSITION 2. *If $C\in\mathcal{A}(n,k)$, then any two of the three assertions* (1), (2), (3) *imply the third.*

PROOF. We use the following known simple identity for the Euler characteristic of C:

$$(*)\qquad \chi(C)=\sum_{i=0}^{k}(-1)^i f_i(C)=\sum_{i=0}^{k}(-1)^i\beta_i(C)\qquad\text{(see e.g., [8, p. 172]).}$$

Note that if $C\in\mathcal{A}(n,k)$ then C is connected (hence $\beta_0(C)=1$), and

$H_i(C)=0$ for all $0<i<k-1$. This follows from the fact that C has a complete $(k-1)$-skeleton. Thus the right hand side of $(*)$ reduces to $1+(-1)^{k-1}\beta_{k-1}(C)+(-1)^k\beta_k(C)$. Note also that

$$\sum_{i=0}^{k-1}(-1)^i f_i(C)=\sum_{i=0}^{k-1}(-1)^i\binom{n}{i+1}=1-(-1)^k\binom{n-1}{k},$$

and therefore $(*)$ reduces to

$$(**)\qquad f_k(C)-\binom{n-1}{k}=\beta_k(C)-\beta_{k-1}(C).$$

Finally note that $H_{k-1}(C)$ is finite iff $\beta_{k-1}(C)=0$, and that $H_k(C)=Z_k(C)$, and therefore $H_k(C)=0$ iff $\beta_k(C)=0$. This, together with $(**)$, clearly implies Proposition 2. ∎

Remark. By Proposition 2, $\mathscr{C}(n,k)$ is precisely the family of k-dimensional **Q**-acyclic complexes, on the vertex set V, with a complete $(k-1)$-skeleton.

Theorem 1.

$$\sum_{C\in\mathscr{C}(n,k)}|H_{k-1}(C)|^2=n^{\binom{n-2}{k}}.$$

Outline of Proof. First we show (Lemmas 1, 2) that for $C\in\mathscr{A}(n,k)$, $\det I_r^k(C)\neq 0$ iff $C\in\mathscr{C}(n,k)$, and that for $C\in\mathscr{C}(n,k)$, $H_{k-1}(C)$ is a finite group and $\det I_r^k(C)=\pm|H_{k-1}(C)|$. We define $M=I_r^k(K)I_r^k(K)^{\mathrm{tr}}$ (K is the complete k-dimensional complex on V). The Cauchy–Binet Theorem implies that

$$\det M=\sum\{(\det I_r^k(C))^2: C\in\mathscr{C}(n,k)\}$$
$$=\sum\{|H_{k-1}(C)|^2: C\in\mathscr{C}(n,k)\}.$$

It remains to show that

$$\det M=n^{\binom{n-2}{k}}.$$

This is done by showing that the eigenvalues of M are 1 and n with multiplicities $\binom{n-2}{k-1}$ and $\binom{n-2}{k}$ respectively.

Now we turn to the detailed proof of Theorem 1.

Lemma 1.

$$\operatorname{rank} I^k(K)=\operatorname{rank} I_r^k(K)=\binom{n-1}{k}.$$

PROOF Each column of $I^k(K)$ is a vector in $B_{k-1}(K)$ (w.r.t. the standard base of $C_{k-1}(K)$) and therefore (by Proposition 1) a linear combination with integer coefficients of the columns corresponding to k-faces that contain x_1. There are $\binom{n-1}{k}$ such columns, and therefore:

$$\operatorname{rank} I_r^k(K) \leqq \operatorname{rank} I^k(K) \leqq \binom{n-1}{k}.$$

But $I_r^k(K)$ restricted to these columns is just a signed permutation matrix: Each $(k-1)$-face not containing x_1 is included in exactly one k-face that contains x_1, and vice versa. Thus $I_r^k(K)$ restricted to these columns is regular and the Lemma follows. ■

LEMMA 2. *Suppose $C \in \mathcal{A}(n,k)$ and $f_k(C) = \binom{n-1}{k}$. Then*:
(1) $\det I_r^k(C) = 0$ *iff* $H_k(C) \neq 0$.
(2) *If* $H_k(C) = 0$ *then* $\det I_r^k(C) = \pm |H_{k-1}(C)|$.

PROOF (1) First note that $H_k(C) = Z_k(C)$ (since $\dim C = k$), and that $\operatorname{rank} I_r^k(C) = \operatorname{rank} I^k(C)$ (by Lemma 1). Define $m = \binom{n-1}{k}$, and let the columns $c_1, \cdots, c_m$ of $I_r^k(C)$ correspond to the k-faces $s_1, \cdots, s_m$ of C. Then $\det I_r^k(C) = 0 \leftrightarrow \operatorname{rank} I_r^k(C) = \operatorname{rank} I^k(C) < m \leftrightarrow$ There are integers $\alpha_1, \cdots, \alpha_m$, not all zero, s.t.

$$\sum_{i=1}^m \alpha_i c_i = 0 \leftrightarrow \sum_{i=1}^m \alpha_i \partial s_i = \partial\left(\sum_{i=1}^m \alpha_i s_i\right) = 0 \leftrightarrow H_k(C) = Z_k(C) \neq 0.$$

(2) $Z_{k-1}(C) = Z_{k-1}(K) = B_{k-1}(K)$ is the submodule of $C_{k-1}(C)$ generated by the columns of $I^k(K)$. $B_{k-1}(C)$ is the submodule of $C_{k-1}(C)$ generated by the columns of $I^k(C)$. Thus we have:

$$H_{k-1}(C) = I^k(K)\mathbf{Z}^{\binom{n}{k+1}} / I^k(C)\mathbf{Z}^{\binom{n-1}{k}}.$$

CLAIM.

$$I^k(K)\mathbf{Z}^{\binom{n}{k+1}} / I^k(C)\mathbf{Z}^{\binom{n-1}{k}} = I_r^k(K)\mathbf{Z}^{\binom{n}{k+1}} / I_r^k(C)\mathbf{Z}^{\binom{n-1}{k}} = \mathbf{Z}^{\binom{n-1}{k}} / I_r^k(C)\mathbf{Z}^{\binom{n-1}{k}}.$$

PROOF OF THE CLAIM. Note that the operation of deleting rows corresponding to faces that contain x_1 induces an isomorphism of $I^k(K)\mathbf{Z}^{\binom{n}{k+1}}$ onto $I_r^k(C)\mathbf{Z}^{\binom{n}{k+1}}$, and this isomorphism maps $I^k(C)\mathbf{Z}^{\binom{n-1}{k}}$ onto $I_r^k(C)\mathbf{Z}^{\binom{n-1}{k}}$. The equality

$$I_r^k(K)\mathbf{Z}^{\binom{n}{k+1}} = \mathbf{Z}^{\binom{n-1}{k}}$$

holds, since $I_r^k(K)$ has a signed permutation submatrix of order $\binom{n-1}{k}$. ■

It follows that

$$H_{k-1}(C) = \mathbf{Z}^{\binom{n-1}{k}} / I_r^k(C)\mathbf{Z}^{\binom{n-1}{k}}.$$

A standard result concerning lattices in $\mathbf{Z}^m$ (see [3, ch. 1]) asserts that if A is an integer valued $m \times m$ matrix with $|\det A| = t \neq 0$, then $\mathbf{Z}^m / A\mathbf{Z}^m$ is a finite group of order t. Therefore in our case $H_{k-1}(C)$ is a finite group of order $|\det I_r^k(C)|$. ∎

Proof of Theorem 1 (cont.). Define $M = I_r^k(K) I_r^k(K)^{tr}$. M is a square matrix of order $\binom{n-1}{k}$. $M(s, \hat{s})$ is the scalar product of the rows of $I_r^k(K)$ that correspond to the $(k-1)$-faces s and $\hat{s}$ of K. Thus

$$M(s,\hat{s}) = \begin{cases} n-k & \text{if } s = \hat{s}, \\ 0 & \text{if } |s \cup \hat{s}| > k+1, \\ I(s, s\cup\hat{s})I(\hat{s}, s\cup\hat{s}) & \text{if } |s\cup\hat{s}| = k+1. \end{cases}$$

Lemma 3. $\det M = n^{\binom{n-1}{k}}$.

Proof Let K^* be the restriction of K to the vertex set $V^* = \{x_2, x_3, \cdots, x_n\}$. K^* is the complete k-dimensional complex over V^*. The proof is based upon the following observations:

(1) $M - I = I^k(K^*) I^k(K^*)^{tr}$.

$I^k(K^*)$ is obtained from $I_r^k(K)$ by deleting the columns of $I_r^k(K)$ that correspond to k-faces of K that contain x_1. The scalar product of two different rows in $I_r^k(K)$ corresponding to the $(k-1)$-faces s and $\hat{s}$ is the same as their product in $I^k(K^*)$, since if $t = s \cap \hat{s}$ is a k-face then $x_1 \notin t$. In every row of $I_r^k(K)$ there is exactly one non-zero entry in columns corresponding to k-faces which contain x_1. Thus in passing from $I_r^k(K)$ to $I^k(K^*)$ the scalar product of every row with itself is reduced by one.

(2) $M - nI = -I^{k-1}(K^*)^{tr} I^{k-1}(K^*)$.

Note that

$$(I^{k-1}(K^*)^{tr} I^{k-1}(K^*))(s,\hat{s}) = \begin{cases} k & \text{if } s = \hat{s}, \\ 0 & \text{if } |s\cap\hat{s}| < k-1, \\ I(s\cap\hat{s}, s)I(s\cap\hat{s}, \hat{s}) & \text{if } |s\cap\hat{s}| = k-1. \end{cases}$$

Thus all the diagonal entries on both sides of (2) are $-k$. Now consider the $(s,\hat{s})$-entry, where s, $\hat{s}$ are distinct $(k-1)$-faces of K^*. If $|s \cup \hat{s}| > k+1$ then $|s \cap \hat{s}| < k-1$, and we get 0 on both sides of (2). If $|s \cup \hat{s}| = k+1$ then $|s \cap \hat{s}| = k-1$, and since $\partial^2(s \cup \hat{s}) = 0$, we have

$$I(s \cap \hat{s}, s)I(s, s \cup \hat{s}) + I(s \cap \hat{s}, \hat{s})I(\hat{s}, s \cup \hat{s}) = 0$$

or, equivalently,

$$I(s, s \cup \hat{s})I(\hat{s}, s \cup \hat{s}) = -I(s \cap \hat{s}, s)I(s \cap \hat{s}, \hat{s}),$$

which is exactly what we need.

Now we return to the proof of the Lemma. From (1) and Lemma 1 we obtain:

$$\operatorname{rank}(M - I) \leqq \operatorname{rank} I^k(K^*) = \binom{n-2}{k}.$$

From (2) and Lemma 1 we obtain:

$$\operatorname{rank}(M - nI) \leqq \operatorname{rank} I^{k-1}(K^*) = \binom{n-2}{k-1}.$$

Thus 1 is an eigenvalue of M with multiplicity m_1,

$$m_1 \geqq \binom{n-1}{k} - \binom{n-2}{k} = \binom{n-2}{k-1}.$$

Similarly, n is an eigenvalue of M with multiplicity m_2,

$$m_2 \geqq \binom{n-1}{k} - \binom{n-2}{k-1} = \binom{n-2}{k}.$$

Since

$$m_1 + m_2 \leqq \binom{n-1}{k},$$

it follows that

$$m_1 = \binom{n-2}{k-1}, \quad m_2 = \binom{n-2}{k}, \quad \text{and} \quad \det M = n^{\binom{n-2}{k}}. \qquad \blacksquare$$

Proof of Theorem 1 (end). The Cauchy–Binet Theorem (see [4, p. 9]) states that if R and S are matrices of respective sizes $p \times q$ and $q \times p$ with $p \leqq q$, then $\det(RS) = \Sigma \det(B)\det(C)$, where the sum extends over all $p \times p$ square submatrices B of R and C of S such that the columns of R in B are numbered the same as the rows of S in C. Thus we have, by Lemma 3 and Lemma 2,

$$n^{\binom{n-2}{k}} = \det M$$
$$= \Sigma \left\{ \det (I_r^k(C) I_r^k(C)^{\mathrm{tr}}) : C \in \mathcal{A}(n,k), f_k(C) = \binom{n-1}{k} \right\}$$
$$= \Sigma \left\{ |H_{k-1}(C)|^2 : C \in \mathcal{A}(n,k), f_k(C) = \binom{n-1}{k}, H_k(C) = 0 \right\}$$
$$= \sum_{C \in \mathscr{C}(n,k)} |H_{k-1}(C)|^2. \qquad \blacksquare$$

4. Further high-dimensional enumeration theorems

Further results concerning tree enumeration can also be generalized to high dimensions. The following examples may serve as an illustration. Associate with each face t of K (= the complete k-complex on $\{x_1, \cdots, x_n\}$) a variable e_t. We assume that all these variables commute, and write e_j for $e_{\{x_j\}}$. Define, for $C \in \mathscr{C}(n,k)$, $\Pi(C) = \Pi\{e_t : t \in C, \dim t = k\}$. Let G be a member of $\mathcal{A}(n,k)$ with $f_k(G) = b$. For a k-face t of G define $Y(t,t) = e_t$. We regard $Y(= Y(G))$ as a diagonal matrix of order b. The following result is an immediate consequence of Lemma 2 and the Cauchy–Binet Theorem.

THEOREM 2. *For any* $G \in \mathcal{A}(n,k)$

(4.1) $\det(I_r^k(G) Y(G) I_r^k(G)^{\mathrm{tr}}) = \Sigma\{ |H_{k-1}(C)|^2 \Pi(C) : C \in \mathscr{C}(n,k), C \subset G\}$. $\blacksquare$

Now we pass to Theorem 3 from Section 1. For $C \in \mathcal{A}(n,k)$ define $\Pi_0(C) = \Pi_{i=0}^n e_i^{\deg_C x_i}$. Theorem 3 is clearly equivalent to

THEOREM 3′. *Let*

$$m_1 = \binom{n-2}{k-1}, \qquad m_2 = \binom{n-2}{k}.$$

Then

$$\sum_{c \in \mathscr{C}(n,k)} |H_{k-1}(C)|^2 \Pi_0(C) = (e_1 + e_2 + \cdots + e_n)^{m_2} \prod_{i=1}^n e_i^{m_1}. \tag{4.2}$$

SKETCH OF PROOF OF THEOREM 3′. Denote the left hand side of (4.2) by A, and the right hand side by B. For a face u define $\pi_0(u) = \Pi\{e_v : x_v \in u\}$, and put $d(u) = \pi_0(u)^{1/2}$. For two p-faces u and w which do not contain x_1, define

$$D_p(u,w) = \begin{cases} d(u) & \text{if } u = w, \\ 0 & \text{if } u \neq w. \end{cases}$$

We regard D_p as a diagonal matrix of order $\binom{n-1}{p+1}$.

Apply Theorem 2 with $G = K$ and with the substitution $e_t \to \pi_0(t)$. This substitution yields A on the right hand side of (4.1). On the left hand side of (4.1) we obtain a determinant that can be rewritten as $\det(D_{k-1}\hat{M}D_{k-1})$, where $\hat{M}(s,s) = \Sigma\{e_\nu : x_\nu \not\in s\}$, and $\hat{M}(s,s') = M(s,s')d(s\Delta s')$ for $s \neq s'$. (Here $M = I_r^k(K)I_r^k(K)^{\mathrm{tr}}$, as in the proof of Theorem 1.) It remains to show that $\det(D_{k-1}\hat{M}D_{k-1}) = B$, or equivalently, that $\det \hat{M} = e_1^{m_1}(e_1 + e_2 + \cdots + e_n)^{m_2}$ (since $(\det D_{k-1})^2 = \Pi_{i=2}^n e_i^{m_1}$).

As in the proof of Lemma 3, it suffices to show that $\operatorname{rank}(\hat{M} - e_1 I) \leqq m_2$ and $\operatorname{rank}(\hat{M} - (e_1 + \cdots + e_n)I) \leqq m_1$. This is done by the following observations:

(1) $\hat{M} - e_1 I = JJ^{\mathrm{tr}}$, where $J(s,t) = d^{-1}(s)I^k(K^*)(s,t)d(t)$. Here $K^* = K \setminus x_1$, as in the proof of Lemma 3, s ranges over all $(k-1)$-faces of K^*, and t ranges over all k-faces of K^*.

(2) $\hat{M} - (e_1 + \cdots + e_n)I = -\hat{J}^{\mathrm{tr}}\hat{J}$, where $\hat{J}(r,s) = d^{-1}(r)I^{k-1}(K^*)(r,s)d(s)$. (Here r ranges over all $(k-2)$-faces of K^*.) ∎

For $C \in \mathscr{A}(n,k)$ define:

$$\Pi_p(C) = \Pi\{e_u^{\deg_C u} : u \in C, \dim u = p\}.$$

PROBLEM 1. Find an explicit evaluation of $\Sigma_{C\in\mathscr{C}(n,k)} | H_{k-1}(C)|^2 \Pi_p(C)$ for $p > 0$.

5. An application

Here we shall prove as a corollary of Theorem 1, that **Q**-acyclic complexes in $\mathscr{A}(n,k)$ (i.e., members of $\mathscr{C}(n,k)$) are, on the average, far from being acyclic.

THEOREM 4. *For fixed $k > 1$,*

(1) *If $C \in \mathscr{C}(n,k)$ then*

$$| H_{k-1}(C)|^2 \leqq (k+1)^{\binom{n-2}{k}}.$$

(2) *For sufficiently large n (i.e., $n > n_0(k)$), the expectation of $| H_{k-1}(C)|^2$ over all members of $\mathscr{C}(n,k)$ satisfies:*

$$\mathrm{E}(| H_{k-1}(C)|^2 : C \in \mathscr{C}(n,k)) > \left(\frac{k+1}{2.8}\right)^{\binom{n-2}{k}}.$$

PROOF. (1) Suppose $C \in \mathscr{C}(n,k)$, and let t be a k-face of C. The number of nonzero entries in the column of $I_r^k(C)$ that corresponds to t is $k+1$ if $x_1 \not\in s$, or one if $x_1 \in s$. Note that

$$\deg_C x_1 \geqq \binom{n-2}{k-1}.$$

(This follows from Theorem 3.) Therefore at most $\binom{n-2}{k}$ k-faces of C miss x_1. It follows by Lemma 2 and Hadamard's inequality, that

$$|H_{k-1}(C)| = |\det(I_r^k(C))| \leqq \sqrt{k+1}^{\binom{n-2}{k}}.$$

(2) Define

$$\mathcal{A}_0(n,k) = \left\{ C \in \mathcal{A}(n,k) : f_k(C) = \binom{n-1}{k} \right\}. \tag{5.1}$$

By Theorem 1 the average of $|H_{k-1}(C)|^2$ over $\mathcal{C}(n,k)$ is

$$n^{\binom{n-2}{k}} \cdot |\mathcal{C}(n,k)|^{-1} \geqq n^{\binom{n-2}{k}} \cdot |\mathcal{A}_0(n,k)|^{-1}$$

$$= n^{\binom{n-2}{k}} \cdot \left(\binom{\binom{n}{k+1}}{\binom{n-1}{k}} \right)^{-1} \geqq n^{\binom{n-2}{k}} \cdot \binom{n-1}{k}! \cdot \binom{n}{k+1}^{-\binom{n-1}{k}}.$$

By Stirling's Formula, we can continue:

$$\geqq n^{\binom{n-2}{k}} \cdot \binom{n-1}{k}^{\binom{n-1}{k}} \cdot e^{-\binom{n-1}{k}} \cdot \binom{n}{k+1}^{-\binom{n-1}{k}} = n^{\binom{n-2}{k}} \cdot \left(\frac{k+1}{en}\right)^{\binom{n-1}{k}}$$

$$= \left(\frac{k+1}{e}\right)^{\binom{n-1}{k}} \cdot n^{-\binom{n-2}{k-1}} = \left(\frac{k+1}{e}\right)^{\binom{n-2}{k}} \cdot \left(\frac{k+1}{en}\right)^{\binom{n-2}{k-1}}.$$

One can easily check that the last expression is

$$> \left(\frac{k+1}{2.8}\right)^{\binom{n-2}{k}}$$

for n large. ■

We conclude this section with some problems concerning possible estimations of the number of complexes in $\mathcal{C}(n,k)$ and in some related families of complexes.

Consider the following subfamilies of $\mathcal{A}_0(n,k)$ (see (5.1)):

$\mathcal{C}_0(n,k)$ — acyclic complexes in $\mathcal{A}(n,k)$.

$\mathcal{D}(n,k)$ — collapsible complexes in $\mathcal{A}(n,k)$.

$\mathcal{B}(n,k)$: A complex C in $\mathcal{A}_0(n,k)$ belongs to $\mathcal{B}(n,k)$ if C does not contain the boundary of a $(k+1)$-simplex.

Obviously $\mathcal{D}(n,k) \subset \mathcal{C}_0(n,k) \subset \mathcal{C}(n,k) \subset \mathcal{B}(n,k) \subset \mathcal{A}(n,k)$.

Problem 2. Estimate $|\mathcal{C}(n,k)|$, $|\mathcal{C}_0(n,k)|$, and $|\mathcal{D}(n,k)|$.

Let $k > 1$ be fixed. We conjecture that for sufficiently large n most complexes in $\mathcal{B}(n, k)$ are also in $\mathcal{C}(n, k)$. We also conjecture that for n large enough most complexes in $\mathcal{C}_0(n, k)$ are not in $\mathcal{D}(n, k)$. (Note that the average degree of a $(k-1)$-face in $\mathcal{C}(n, k)$ is $(k+1)(n-k)/n$.)

6. Duality

It follows from Theorem 1 that

$$\sum_{C \in \mathcal{C}(n+2,k)} |\hat{H}_{k-1}(C)|^2 = (n+2)^{\binom{n}{k}} = \sum_{C \in \mathcal{C}(n+2,n-k)} |\hat{H}_{n-k-1}(C)|^2 \qquad \text{for } 0 < k < n.$$

As we shall see, there is a natural bijection $C \to B(C)$ between $\mathcal{C}(n+2, k)$ and $\mathcal{C}(n+2, n-k)$ such that $\hat{H}_{k-1}(C) \approx \hat{H}_{n-k-1}(B(C))$. (Here we use the reduced homology $\hat{H}_*(C)$, so as to include the values $k = 1$ and $k = n-1$.)

Let $V = \{x_1, \cdots, x_{n+2}\}$ be a fixed vertex set. For a simplicial complex C on the vertex set V, define

$$B(C) = \{S \subset V : V \setminus S \notin C\}.$$

$B(C)$ is sometimes called the *blocker* of C.

The reader will easily verify the following properties of $B(C)$.

1. $B(C)$ is a simplicial complex.
2. $C \subset C'$ iff $B(C') \subset B(C)$.
3. $B(B(C)) = C$.
4. $f_j(C) + f_{n-j}(B(C)) = \binom{n+2}{j+1}$.
5. C is isomorphic to C' iff $B(C)$ is isomorphic to $B(C')$.
6. $C \in \mathcal{A}(n+2, k)$ iff $B(C) \in \mathcal{A}(n+2, n-k)$.
7. C collapses to C' iff $B(C')$ collapses to $B(C)$.

THEOREM 5. (a) $C \in \mathcal{C}(n+2, k)$ *iff* $B(C) \in \mathcal{C}(n+2, n-k)$.
(b) *For* $C \in \mathcal{C}(n+2, k)$, $\hat{H}_{k-1}(C) \approx \hat{H}_{n-k-1}(B(C))$.

PROOF (sketch). It follows from the Alexander Duality Theorem (see [6, ch. 5] or [1, vol. 3 pp. 24–26] for a suitable version of the Alexander Duality Theorem), that for a simplicial complex C

$$\hat{H}_{k-1}(C) \approx \hat{H}^{n-k}(B(C)).$$

Let $\hat{H}_i(C) = F_i(C) \oplus T_i(C)$, $\hat{H}^i(C) = F^i(C) \oplus T^i(C)$, where $F_i(C)$ $[F^i(C)]$ and $T_i(C)$ $[T^i(C)]$ are respectively the free part and the torsion of $\hat{H}_i(C)$ $[\hat{H}^i(C)]$. It is known (see [6, p. 168] or [8, pp. 241–244]) that

$$F^i(C) \approx F_i(C) \quad \text{and} \quad T^i(C) \approx T_{i-1}(C).$$

It follows that $C \in \mathscr{C}(n+2,k)$ iff $B(C) \in \mathscr{C}(n+2,n-k)$, and that for $C \in \mathscr{C}(n+2,k)$

$$\hat{H}_{k-1}(C) \approx \hat{H}^{n-k}(B(C)) \approx T^{n-k}(B(C)) \approx T_{n-k-1}(B(C)) \approx \hat{H}_{n-k-1}(B(C)). \quad \blacksquare$$

Finally note that if $C \in \mathscr{C}(n+2,k;\, m+d_1,\cdots,m+d_{n+2})$, then

$$B(C) \in \mathscr{C}(n+2,n-k;\, m'+d_1,\cdots,m'+d_{n+2}),$$

$$\text{where} \quad m = \binom{n}{k-1} \quad \text{and} \quad m' = \binom{n}{n-k-1}.$$

7. Some examples

In this section we describe in some detail the families $\mathscr{C}(n,2)$ for $n \leqq 6$.

If $C \in \mathscr{C}(n,2)$ and x is a vertex of C, then link (x, C) is a graph. Using the fact that $H_1(C, \mathbf{Q}) = 0$, it is easy to show that link (x, C) is connected for every vertex x.

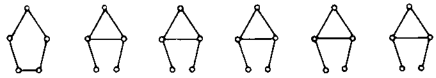
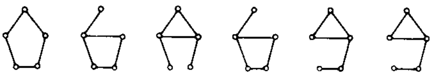
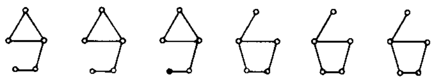
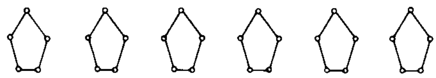
$\mathscr{C}(3,2)$ consists of a single complex — a triangle. $\mathscr{C}(4,2)$ consists of the four isomorphic complexes obtained by taking three of the 2-faces of the simplex on four vertices. $\mathscr{C}(5,2)$ consists of three isomorphism types of complexes which are dual (in the sense of Section 6) to the three isomorphism types of trees on 5 vertices. All these complexes are collapsible.

For $C \in \mathscr{C}(6,2)$, if some vertex z has degree 4, then link (z, C) is a tree, and therefore the faces that contain this vertex can be collapsed one by one, and the whole complex collapses to a complex in $\mathscr{C}(5,2)$ and thus is collapsible. There are altogether 45,936 ($=6^6-720$) such complexes. The remaining complexes, with all vertices of degree 5, fall into four isomorphism types (see Table 1). Three of them, with automorphism groups of orders 2, 3 and 10 respectively (altogether 672 complexes), are again collapsible and are counted in Theorem 1 just once. The last isomorphism type contains the triangulations of the projective plane P^2 that are obtained by identifying opposite faces of a regular isocahedron. Here the automorphism group is of order 60, and therefore there are 12 different triangulations. Since $H_1(P^2) = \mathbf{Z}_2$, each such complex is counted four times in Theorem 1.

Finally we would like to draw attention to the subfamily of self-dual complexes in $\mathscr{C}(6,2)$. $C \in \mathscr{A}(2n, n-1)$ is self dual if $B(C) = C$ or equivalently if $s \cap t \neq \varnothing$ for every two $(n-1)$-faces s and t of C.

There are 4 isomorphism types of self-dual complexes in $\mathscr{C}(6,2)$ (see Table 2).

Table 1. Complexes in $\mathscr{C}(6,2)$ with degree sequence (5, 5, 5, 5, 5, 5)

C	2-faces	link of 1 2 3 4 5 6	Aut(C)	$H_1(C)$	*
C_1	123 134 145 156 126 234 345 456 256 236		D_{10}	1	72
C_2	123 134 145 156 126 234 235 256 346 456		Z_2	1	360
C_3	124 125 134 145 136 235 236 256 346 356		Z_3	1	240
P_2	123 134 145 156 126 235 346 245 356 246		A_5	Z_2	12

* Number of complexes in $\mathscr{C}(6,2)$ isomorphic to C ($=720/|\mathrm{Aut}(C)|$).

Table 2. Self-dual complexes in $\mathscr{C}(6,2)$

C	degrees	2-faces	Aut(C)	*
D_1	10, 4, 4, 4, 4, 4	123 124 125 126 134 135 136 145 146 156	S_5	6
D_2	8, 6, 4, 4, 4, 4	123 124 125 126 134 135 146 156 245 236	$(Z_2)^3$	90
D_3	6, 6, 6, 4, 4, 4	123 124 134 125 136 156 235 236 246 345	S_3	120
P_2	5, 5, 5, 5, 5, 5	See Table 1		

* Number of complexes in $\mathscr{C}(6,2)$ isomorphic to C.

The total number of self-dual complexes which are not isomorphic to P_2 is 6^3 ($=\sqrt{6^6}$). Moreover

$$\Sigma\{|H_1(C)|^2\Pi_0(C): C\in\mathscr{C}(6,2),\ B(C)=C,\ C\not\simeq P^2\}=(e_1^2+e_2^2+\cdots+e_6^2)^3\prod_{i=1}^{6}e_i^4.$$

A trivial check shows a similar phenomenon in $\mathscr{C}(4,1)$.

PROBLEM 3. Extend the above observation to self-dual complexes in $\mathscr{C}(2n,n-1)$. (For a related result concerning enumeration of trees see [9].)

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Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel